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The motion of a cavity in a vertical rotating tube

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Experiments are performed to measure the upward velocity U of the cavity resulting from the draining of a vertical liquid-filled tube of radius b, which rotates about its axis at various speeds. With no rotation the cavity moves so that the Froude number $U/(gb)^{\frac{1}{2}}$ is constant, where g is the gravitational acceleration. For high angular velocities Ω, however, the cavity appears to adopt a constant value of the Rossby number $U/\Omega b$, implying that gravitational forces are dominated by centrifugal forces. The cavity velocity finally achieved is found to be the same as the maximum group velocity of infinitesimal waves, so that $U/\Omega b = 0{\cdot}52$. Approximate theoretical models which satisfactorily describe the development of the motion are constructed.

1. Introduction

In the experiments reported in this paper, a long tube of radius b filled with water and held with its axis vertical was rotated steadily about its axis at various speeds, denoted by Ω. After the water had acquired rigid-body rotation the tube was opened at its lower end and an axisymmetric air-filled cavity then propagated in the tube with velocity U while the water drained from the open end. These experiments form part of a programme of research into the motion of gas slugs in various flow situations and for the remainder of the paper the cavity will be termed a slug as is usual in non-rotating systems. In the absence of rotation the gravitational force will determine the motion and the system will operate at some constant value of Froude number $U/(gb)^{\frac{1}{2}}$, where g is the acceleration due to gravity. At high values of Ω, however, it is anticipated that centrifugal forces will be dominant so that the motion will be characterized by some constant value of the Rossby number $U/\Omega b$. We are interested here in studying the transition between these two different regimes.

For high values of Ω, experiments in a related system have been carried out by Benjamin & Barnard (1964). In their experiments, which were conceived as a possible laboratory demonstration of the vortex breakdown phenomenon, the tube was horizontal and rotational speeds were arranged so that the ratio $\Omega^2 b/g$ took values from 41 to 138. These were presumed to be high enough to ensure that centrifugal forces predominated over gravitational forces so that the motion would be approximately axisymmetric, and it was found that the Rossby number was sensibly constant at a value of approximately $0{\cdot}38$. The radius of the slug some way downstream from its nose was found to be approximately $\frac{1}{2}b$ and the slug generally exhibited waves on its surface which were thought to be an essential feature of the flow and not caused by any vibrations of the apparatus.

Although they had no prediction of the numerical value of Rossby number which would be achieved, Benjamin & Barnard at first expected to observe a flow, designated 'type A', which would be steady in a frame of reference moving with the slug and undisturbed far upstream. In fact, their experiments demonstrated the existence of the Taylor phenomenon, which they called a flow of type B, in which a continually lengthening column of fluid moved ahead of the cavity and would thus ultimately interfere with the uniformity of the flow far upstream. The velocity of propagation of this column was found experimentally to be about $0{\cdot}92c_0 = 0{\cdot}48\Omega b$, where c_0 is the theoretical propagation velocity of the front – that is, the maximum group velocity of infinitesimal waves. Experimentally the slug velocity was found to be $U = 0{\cdot}73c_0$. During their investigation Benjamin & Barnard discovered a theoretical argument which demonstrated that if gravity is neglected a mathematical solution of an idealized model of the flow past a constant-pressure slug cannot be of type A. This result was confirmed by rigorous proof in an appendix to their paper by Fraenkel.

For high values of Ω it is natural to compare results from the present experiments with those of Benjamin & Barnard but it must be stressed that the two systems are essentially different in view of the different attitudes to the gravitational field. In the present arrangement, for example, the cavity is axisymmetric for all values of Ω whereas in Benjamin & Barnard's situation, $\Omega^2 b/g$ was necessarily large in order to make the motion approximately axisymmetric. Consider the general situation in which the tube is inclined at an angle α to the downward vertical direction. If surface tension and viscous forces are considered unimportant then dimensional analysis suggests that

$$f_1[U/(gb)^{\frac{1}{2}}, U/\Omega b, \alpha] = 0. \tag{1}$$

Alternative forms which involve bubble velocity in only one of the groups are

$$U/(gb)^{\frac{1}{2}} = f_2[\Omega^2 b/g, \alpha], \tag{2}$$

or

$$U/\Omega b = f_3[\Omega^2 b/g, \alpha]. \tag{3}$$

In these forms the dimensionless group $\Omega^2 b/g$ expresses the ratio of centrifugal to gravitational forces and this is the parameter which is varied in the experiments. Values of α in the range $\frac{1}{2}\pi < \alpha < \pi$ correspond to situations in which the tube would be opened at its upper end and in these cases, for finite values of Ω, the slug can penetrate only a finite distance from the mouth of the tube; with $\alpha = \pi$, for example, this distance would be $\Omega^2 b^2/2g$. In the range $0 < \alpha < \frac{1}{2}\pi$, however, the slug propagates indefinitely for all values of $\Omega^2 b/g$. This consideration emphasizes the fact that α is a pertinent parameter; the present experiments relate to the condition $\alpha = 0$, while Benjamin & Barnard's experiments had $\alpha = \frac{1}{2}\pi$.

In the absence of rotation it is anticipated that the Froude number $U/(gb)^{\frac{1}{2}}$ will adopt a value which is constant for a given α and which will be denoted here by $f_2(0, \alpha)$. Dumitrescu's (1943) experiments showed that $f_2(0, 0) = 0{\cdot}49$, in good agreement with his own theory, while Zukoski's (1966) experiments gave

$f_2(0, \frac{1}{2}\pi) = 0{\cdot}75$, which agrees well with Benjamin's (1968) theory for this situation. When $\Omega^2 b/g$ is moderately large, it is reasonable to anticipate that the effect of gravity will be dominated by that due to centrifugal forces and that the Rossby number $U/\Omega b = f_3(\Omega^2 b/g, \alpha)$ will then adopt a value which is almost constant for a given α. The present experiments, which are described in § 2, do show that $f_3(\Omega^2 b/g, 0)$ varied very little from a value $0{\cdot}52$ over the range $12 \leqslant \Omega^2 b/g \leqslant 27$. Benjamin & Barnard interpreted their own experimental results, which covered the range $41 \leqslant \Omega^2 b/g \leqslant 138$, as showing $f_3(\Omega^2 b/g, \frac{1}{2}\pi)$ to be essentially constant with a value of about $0{\cdot}38$.† Whether or not

$$f_3(\infty, 0) = f_3(\infty, \tfrac{1}{2}\pi)$$

is left as an open question for the present.

Benjamin & Barnard's and Fraenkel's proofs that a steady motion relative to the slug is theoretically impossible are restricted in their applicability by the requirement that velocities are finite everywhere, and the implications of this restriction may be seen by considering their arguments. Benjamin & Barnard's theoretical model is of an axisymmetric flow and they considered a momentum balance between the cylindrical flow far upstream of the nose of the slug and an assumed cylindrical flow in the annular space far downstream. If the gravitational acceleration is directed along the tube axis, however, velocities will become unbounded sufficiently far downstream, the flow will be falling freely there and velocities on the slug surface will vary as $(gx)^{\frac{1}{2}}$, where x is the distance downstream from the stagnation point. The assumption of a cylindrical flow in that region is then untenable. Moreover, since $\alpha = 0$ in their model, the motion can be truly axisymmetric only if $\Omega^2 b/g \rightarrow \infty$, with Ω presumably finite. It appears that the flow postulated corresponds to the situation $g = 0$ and the proof of non-existence of the steady flow of type A is thus restricted to the situation at the limit $\Omega^2 b/g = \infty$, with $g = 0$. At this limit Benjamin & Barnard showed that $Ro \ngtr 0{\cdot}52$, but the actual value was not determined theoretically.

Fraenkel's proof is similarly restricted. He did remove the assumption that the flow is cylindrical far downstream, thus including the possibility of wavy slug surfaces but the restriction to finite velocities remains. His proof takes the form of a *reductio ad absurdum* on this very point. We note that this definitive proof of non-existence for the condition $g = 0$ does not preclude the construction of a theory applicable to the present experimental situation. Although the gravitational forces may be dominated by centrifugal forces by making Ω sufficiently large, the gravitational force is non-zero and is directed along the tube axis throughout the present experiments. This feature is accordingly represented in the theory of § 3, where it is assumed that $g \neq 0$ so that the proofs of non-existence do not then apply. It will be apparent that we should avoid the statement that gravity is negligible at large values of $\Omega^2 b/g$ since this implies that we may take $g = 0$. It is more appropriate to describe the influence of gravity as insignificant in the sense of being of secondary importance although not negligible.

† See figure 6 below for a summary of both sets of results.

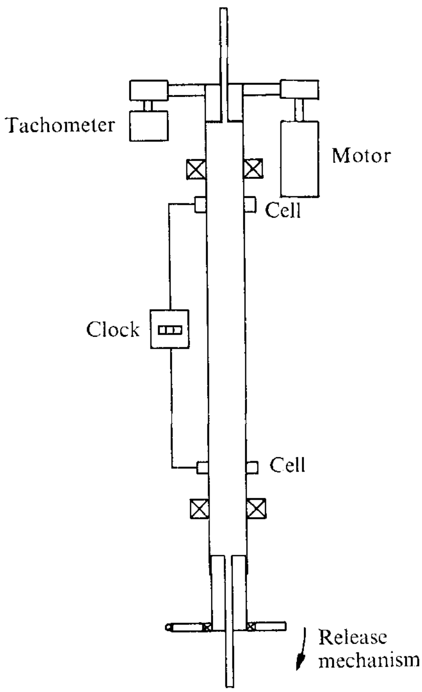

FIGURE 1. Schematic arrangement of experimental apparatus.

2. Experiments

Figure 1 shows schematically the arrangement of the apparatus. A perspex tube of internal diameter 65 mm, wall thickness 4·5 mm and length 1·865 m was supported in bearings on a rigid channel section and the whole was set vertically against a wall. The tube was driven by an electric motor equipped with a continuously variable gear box through a system of pulleys which enabled speeds to range up to 900 rev/min. The hollow core of the driving spindle to which these pulleys were attached communicated with the interior of the perspex tube, and a filter pump could be attached to the end of the spindle thus allowing water to be drawn into the tube via a rubber hose from a tank below. At its lower end the perspex tube was sealed with a removable stopper which was free to rotate with the tube. When filled, the tube was sealed at its two ends, the water and pump connexions were removed and the tube rotated at the desired speed. Several minutes were allowed to elapse so that the water in the tube had acquired a rigid-body rotation by viscous action before the slug was formed by removing the stopper. Particularly at high rotational speeds, it was found advantageous to open the needle valve at the upper end very slightly so that a minute quantity of air could be drawn into the spindle while the lower stopper was being removed. This prevented the formation of small cavitation bubbles on the tube axis.

The rotational speed of the tube was measured with a tachometer consisting of a small electric generator run from the driving spindle, the speed being continuously displayed on a meter. During its passage along the tube the nose of the slug interrupted two narrow light beams which were directed across a diameter

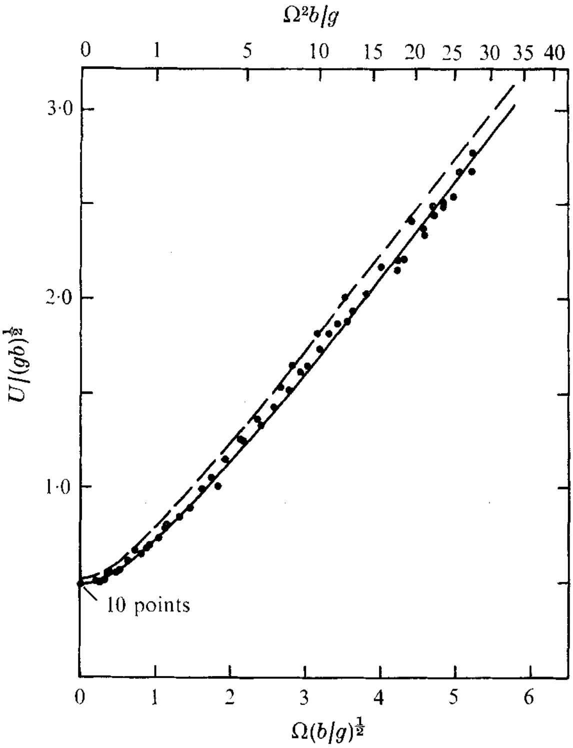

FIGURE 2. Variation of Froude number with $\Omega(b/g)^{\frac{1}{2}}$. – – –, single-term first approximation; ——, three-term second approximation.

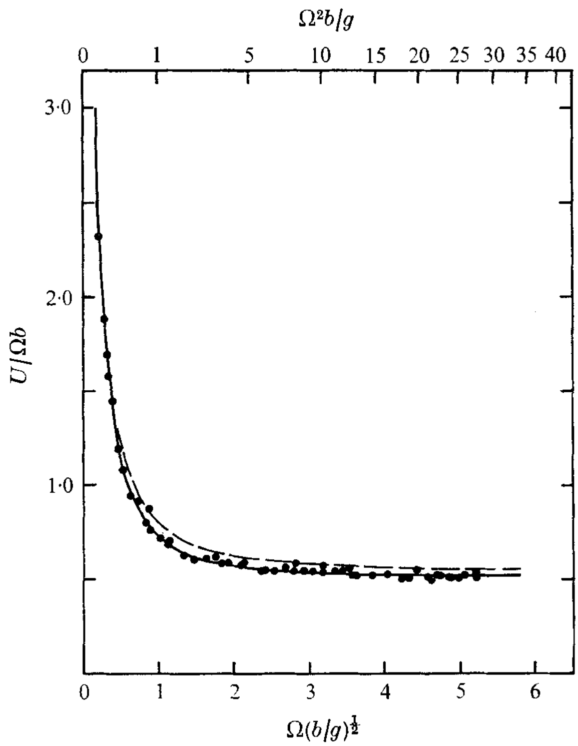

FIGURE 3. Variation of Rossby number with $\Omega(b/g)^{\frac{1}{2}}$. – – –, single-term first approximation; ——, three-term second approximation.

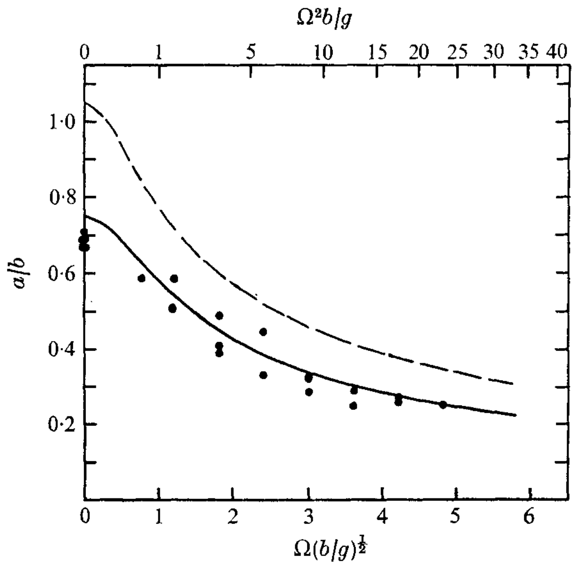

FIGURE 4. Variation of nose curvature with $\Omega(b/g)^{\frac{1}{2}}$. – – –, single-term first approximation; ——, three-term second approximation. (The theoretical lines show curvature at the stagnation point; the measured values are mean values measured over a finite arc.)

of the tube and which fell on two photocells fixed a known distance apart. The signals generated in this way were made to start and stop a digital clock, and the slug velocity was thus determined. Photographs were taken with a 35 mm reflex camera which viewed the slugs against a diffusing screen illuminated from behind. From the negatives of these photographs the radius of curvature of the nose of a slug was measured by projecting an enlarged image on to a screen so as to allow comparison with a set of circular arcs. In order to correct for the distortion introduced by the water-filled tube, photographs were also taken of a set of concentric circles inscribed on a thin sheet of perspex which was fixed within the tube in an axial plane and illuminated in the same fashion as were the slugs. In this way it was found that a true radius of curvature a appeared as an image on the negative whose apparent radius of curvature at the axis was $1{\cdot}63a$.

Measurements of velocity were made for over sixty slugs at various rotational speeds; approximately twenty were photographed for curvature measurement. The results of the experimental measurements of velocity plotted in dimensionless form† are shown in figures 2 and 3. Figure 4 and figure 5 (plate 1) show the radii of curvature, denoted by a, and the general development of slug shape with increasing rotational speed. The lines shown on these figures come from approximate theoretical models described in § 3. With no rotation, as expected, the present results are in good agreement with those of previous investigators. The mean value of Froude number obtained here, $Fr = 0{\cdot}49$, was also determined by Dumitrescu (1943); the present nose radius of curvature, $a = 0{\cdot}69b$, compares

† $\Omega(b/g)^{\frac{1}{2}}$ has been used as abscissa in most cases because this gives a less constricted view of the changes which occur when $\Omega^2 b/g < 1$. A secondary axis shows the force ratio $\Omega^2 b/g$.

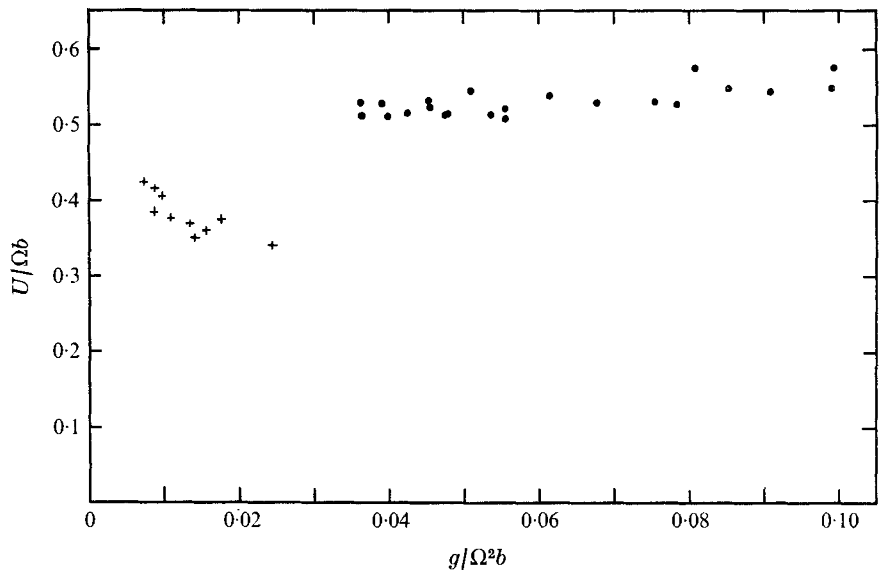

FIGURE 6. Variation of Rossby number with $g/\Omega^2 b$. ●, present experiments, $\alpha = 0$; +, Benjamin & Barnard's experiments, $\alpha = \frac{1}{2}\pi$.

well with Dumitrescu's value $a = 0{\cdot}71b$. As $\Omega^2 b/g$ increases, the Froude number rises quite rapidly while the Rossby number falls at a progressively decreasing rate. For values of $12 \leqslant \Omega^2 b/g \leqslant 27$, approximately, figure 2 shows that U appears to vary linearly with Ωb, and figure 3 confirms that the Rossby number is essentially constant over this range at a value $Ro = 0{\cdot}52(3)$. For lower values of $\Omega^2 b/g$, the Rossby number exceeds this value, so that the slug propagates with a velocity greater than the maximum group velocity of infinitesimal waves, which is $c_0 = 0{\cdot}52\Omega b$. There is thus no upstream influence in this system and the flow upstream relative to the slug remains steady. It is interesting to compare these results with those of Benjamin & Barnard, who found that, in the horizontal tube for $41 \leqslant \Omega^2 b/g \leqslant 138$, the Rossby number was sensibly constant at a value $Ro = 0{\cdot}38$. Upstream influence was detected in their experiments, the slug exhibited wave formations on its surface, and in the photograph of a slug which they printed, the motion was only approximately axisymmetric. The most notable features of figure 5 (plate 1), on the other hand, are the absence of any wave formations and the very high degree of symmetry. Figures 4 and 5 show that the nose of the slug becomes progressively sharper as $\Omega^2 b/g$ is increased.

It has been suggested by a referee that the two systems should approach a common asymptotic value. In figure 6 the data available have been plotted against $g/\Omega^2 b$ in order that the trend of experimental points in approaching the limit $g/\Omega^2 b \to 0$ may be tentatively assessed. For both systems, extrapolation of experimental results would involve an assumption that there are no discontinuities in the behaviour outside the experimental ranges tested. In figure 6 the range over which the present experimental data have been taken to exhibit a constant Rossby number, $Ro = 0{\cdot}52$, is $0{\cdot}035 \leqslant g/\Omega^2 b \leqslant 0{\cdot}08$. Benjamin & Barnard interpreted their data, also shown on this figure, as giving a constant value $Ro = 0{\cdot}38$. It is difficult to draw any firm conclusion on the asymptotic behaviour beyond saying that in neither set of data does the trend conflict with Benjamin & Barnard's theoretical limitation that $Ro \ngtr 0{\cdot}52$ for the condition $g = 0$. More data would be desirable for both systems in the range $g/\Omega^2 b < 0{\cdot}01$ and it is hoped that it will be possible at some later date to modify the apparatus so as to investigate this range in more detail for various values of the incidence angle α. The present objective is to construct an approximate theory to describe the transition from constant Froude number to constant Rossby number behaviour which is exhibited by the present range of experiments.

3. Theory

In view of the absence of upstream influence in the flow the theoretical problem to be considered is far simpler than that which arose from Benjamin & Barnard's experiments. The problem is to construct a solution representing the inviscid axisymmetric flow of a steady swirling stream past a gas slug in which the pressure is constant. The gravitational field acts along the tube axis and, thus, the boundary condition of constant pressure on the slug surface requires the fluid to accelerate continuously under the influence of gravity away from the stagnation point at the nose. For the horizontal tube this boundary condition would require the fluid to be stagnant relative to the slug everywhere on its surface.

The origin of a set of cylindrical polar co-ordinates (x, r, ϕ) is taken at the nose of the slug on the centre-line of the tube and moving with the slug at velocity U. The positive direction of x is vertically downwards and local particle velocities in the x and r directions are denoted by u and v respectively. Relative to this non-rotating co-ordinate system the local angular velocity $\omega = w/r$, where w is the azimuthal component of velocity, and the tube rotates with constant angular velocity Ω. Since the flow is axisymmetric it may be described in terms of Stokes's stream function ψ, so that $ru = \partial\psi/\partial r$ and $rv = -\partial\psi/\partial x$. From Batchelor (1967) the equation governing ψ when the flow some distance upstream has rigid-body rotation Ω and uniform axial velocity U (so that $\psi = \frac{1}{2}Ur^2$ far upstream) is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{2\Omega^2}{U}r^2-\frac{4\Omega^2}{U^2}\psi. \tag{4}$$

Further, on modifying Batchelor's equations to include the gravitational term, from Bernoulli's theorem for this situation we have

$$\tfrac{1}{2}(u^2+v^2+w^2)+\frac{p}{\rho}-gx = H(\psi) = \tfrac{1}{2}U^2+\frac{2\Omega^2}{U}\psi, \tag{5}$$

and from the constancy of circulation around a circle centred on the axis and normal to it, the azimuthal component of velocity w is given by the relation

$$rw = C(\psi) = (2\Omega/U)\,\psi. \tag{6}$$

The surface of the slug will be the stream surface $\psi = 0$, where, as (6) shows, the azimuthal component of velocity is zero. By virtue of (5) the boundary condition of constant pressure may be written as

$$q^2-2gx = 2(p_0-p)/\rho = 0 \quad \text{on} \quad \psi = 0, \tag{7}$$

where $q^2 = u^2+v^2$ and p_0 is the stagnation pressure at the origin.

Solutions of (4) which are applicable to problems in which the fluid is contained in a cylindrical vessel of radius b, as in the present problem, have been given by Squire (1956). These are used to construct a solution of the form

$$\psi = U\left(\tfrac{1}{2}r^2 - rb\sum_{i=1}^{\infty}\frac{d_i}{\gamma_i}\exp(\alpha_i x/b)\,J_1(\gamma_i r/b)\right), \tag{8}$$

in which the quantities d_i are coefficients whose values are to be determined, γ_i is the ith zero of the first-order Bessel function J_1, and where

$$\alpha_i^2 = \gamma_i^2 - (2\Omega b/U)^2. \tag{9}$$

In constructing such a solution it is assumed on the basis of our experimental evidence that $\alpha_1^2 > 0$ so that $Ro > 2/\gamma_1 = 0{\cdot}52$. This restriction ensures the absence of upstream influence and it is known that in this range of Rossby number the general character of the swirling flow resembles that without swirl (Squire 1956; Fraenkel 1956). Indeed for the case when $\Omega = 0$, where $\alpha_i = \gamma_i$, equation (8) reproduces those solutions proposed by Davies & Taylor (1950), Dumitrescu (1943) and Layzer (1955) for the non-rotating problem. The approximate solutions subsequently obtained in this paper may be seen as straightforward extensions of solutions which have already been obtained for the non-rotating system.

We may comment here that, for arbitrary $\Omega^2 b/g$, questions of the existence and uniqueness of solutions remain open. Garabedian (1957) has pointed out that in the equivalent plane problem without rotation the flows are not uniquely determined by specifying g and the channel dimension b, but that the velocity U may also be specified. It appears that there is an infinite number of solutions and hence slug shapes which will satisfy the boundary condition expressed in (7), and the implication of his work is that a similar indeterminacy exists for the axisymmetric problem without rotation. As has already been seen in § 1, Benjamin & Barnard and Fraenkel have proved that, for the situation $g = 0$, no steady solutions of type A exist which satisfy the constant-pressure boundary condition, which then assumes the rather different form $q = 0$ on $\psi = 0$. In the present theory this latter situation does not arise since g is taken to be non-zero.

The velocity components in the flow described by (8) are

$$u = U\left(1 - \sum_{i=1}^{\infty} d_i \exp(\alpha_i x/b)\,J_0(\gamma_i r/b)\right) \tag{10}$$

and

$$v = U\sum_{i=1}^{\infty}\frac{d_i\alpha_i}{\gamma_i}\exp(\alpha_i x/b)\,J_1(\gamma_i r/b), \tag{11}$$

and since there is a stagnation point at the origin it follows that

$$\sum_{i=1}^{\infty} d_i = 1. \tag{12}$$

The surface of the slug, corresponding to the stream surface $\psi = 0$ in (8), is given by

$$r = 2b\sum_{i=1}^{\infty}\frac{d_i}{\gamma_i}\exp(\alpha_i x/b)\,J_1(\gamma_i r/b), \tag{13}$$

from which the radius of curvature a at the nose is determined as

$$a = 4b\frac{\sum_{i=1}^{\infty} d_i\alpha_i}{\sum_{i=1}^{\infty} d_i\gamma_i^2}. \tag{14}$$

Following methods of solution in non-rotating situations (Dumitrescu 1943; Collins 1965) we expand the left-hand side of (7) in a Taylor series about the origin to give

$$q^2-2gx = [(q^2-2gx)_0'']\frac{r^2}{2!}+[(q^2-2gx)_0^{\text{iv}}]\frac{r^4}{4!}+[(q^2-2gx)_0^{\text{vi}}]\frac{r^6}{6!}+\dots. \tag{15}$$

Here primes denote derivatives with respect to r of a quantity evaluated on $\psi = 0$ and regarded as a function of r only, and the suffix 0 denotes evaluation at the origin. Only derivatives of even order appear in the expansion since q^2 and $-gx$ are even functions of r. The coefficients d_i are then evaluated in principle by determining those values necessary to reduce all terms in (15) to zero. Elimination of the coefficient of r^2 requires that

$$(u^2+v^2)_0'' = 2gx_0'' \tag{16}$$

or

$$[v']_0^2 = g/a, \tag{17}$$

where a is the radius of curvature at the nose of the slug, or

$$U = (ga)^{\frac{1}{2}}/[av'/U]_0. \tag{18}$$

This result, which is exact and which reproduces the non-rotating result (Collins 1965; Batchelor 1967), underlines the role played by the radius of curvature of the slug nose in determining the slug velocity. The quantity av_0'/U is a dimensionless constant dependent only on the slug geometry. With the velocity component v given by (11) and the radius of curvature by (14) the slug Froude number is found to be

$$\frac{U}{(gb)^{\frac{1}{2}}} = \frac{\left(\sum_{i=1}^{\infty} d_i\gamma_i^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{\infty} d_i\alpha_i\right)^{\frac{3}{2}}}, \tag{19}$$

a result which may be used to eliminate the Froude number from subsequent coefficients in (15).

Consider n terms in the series in (8). Equation (12) provides one relationship for the n coefficients d_i and the remaining $n-1$ equations necessary for solution arise by eliminating all coefficients up to and including the coefficient of r^{2n} in (15), employing (18) in the process. With the d_i determined in this way, the slug Froude number, its nose radius of curvature and the slug shape follow from (19), (14) and (13). In principle $n = \infty$ terms would be needed in order to solve the free-boundary problem but the equations which arise for the coefficients d_i (which are functions of Rossby number) are of rapidly increasing complexity as n is increased and simple analytical relationships for the terms d_i are not obtained. In order to proceed some approximation is necessary, and as with the non-rotating approximate solutions it is found that satisfactory descriptions of the experimental data may be obtained using only very small numbers of terms in the series.

The objective of approximate solutions may be seen by reference to equation (18) to be the construction of a suitable model of the slug geometry, particularly in the vicinity of its stagnation point, in order that the dimensionless quantity av_0'/U may be deduced from the model. This was essentially the approach adopted by Davies & Taylor in dealing with spherical-cap bubbles. In fact, just one term of the series in (8) describes a flow past a shape of slug-like character although, as will appear shortly, the radius of curvature at its nose is rather high. This single-term approximation was also used for the non-rotating slug by Davies & Taylor, who chose to satisfy the constant-pressure boundary condition at the point on the slug where $r = \frac{1}{2}b$. The present extension of the single-term approximation to the swirling flow is, however, more closely associated with Layzer's (1955) work on the non-rotating slug since he satisfied the boundary condition at the nose in the same manner as is employed here. Approximations will be identified by the number of terms employed in (8) and by the number of coefficients eliminated in (15). Thus, the 'n-term mth approximation' employs n terms in (8) and eliminates m terms in (15).

On setting $n = 1$, equation (12) shows that $d_1 = 1$. The Froude number for this single-term first approximation then follows from (19) as

$$\frac{U}{(gb)^{\frac{1}{2}}} = \frac{\gamma_1}{[\gamma_1^2-(2\Omega b/U)^2]^{\frac{3}{4}}} \tag{20}$$

and the nose radius of curvature is given by

$$\frac{a}{b} = \frac{4}{\gamma_1^2}[\gamma_1^2-(2\Omega b/U)^2]^{\frac{1}{2}}, \tag{21}$$

with $\gamma_1 = 3{\cdot}8317...$ in both equations. When compared with experiment as shown by the broken lines in figures 2 and 3, this simple approximation is seen to provide a very good description of the range of experimental data recorded in the present experiments. With $\Omega = 0$, equations (20) and (21) reproduce Layzer's results, which are

$$U/(gb)^{\frac{1}{2}} = 0{\cdot}511 \quad \text{with} \quad a/b = 1{\cdot}014. \tag{22}$$

Experimental values for $\Omega = 0$ are $Fr = 0{\cdot}49$ with $a/b = 0{\cdot}69$. At the highest value of Ω^2b/g employed here (that is, $\Omega^2b/g = 27$) equation (20) gives $Ro = 0{\cdot}56$, compared with the experimental value of $Ro = 0{\cdot}52(3)$. The single-term first approximation is seen to conform with the assumption that $Ro > 0{\cdot}52$ over the range of the experiments and indeed for all Ω^2b/g. The limiting behaviour of (20) is that, as $\Omega^2b/g \to \infty$ with $g \neq 0$, $U/(gb)^{\frac{1}{2}} \to \infty$ in a manner such that

$$[U/(gb)^{\frac{1}{2}}]/[\Omega(b/g)^{\frac{1}{2}}] = \frac{U}{\Omega b} \to \frac{2}{\gamma_1} = 0{\cdot}52. \tag{23}$$

(The same result would be obtained by allowing $g \to 0$ in (20).)

As it is derived from an approximate flow pattern this result cannot be claimed to describe the asymptotic behaviour of the exact solution to the free-boundary problem. What can be said is that, provided that the physical situation conforms with the assumptions of the theory, namely $g \neq 0$ and $Ro > 0{\cdot}52$, equation (23) provides a first approximation to the behaviour as $\Omega^2 b/g$ becomes moderately large.

On recalling that the objective is to construct a model of the flow geometry, in particular near the stagnation point, it will be clear from figure 4 that a major deficiency of the single-term first approximation is that its nose radius of curvature is consistently too high, although the general variation with $\Omega^2 b/g$ conforms well with that found experimentally. We now consider the development of Dumitrescu's approximate solution for the non-rotating slug to cover the swirling flow situation. His solution is known to provide a good description of the slug shape and the Froude number and it constitutes a three-term second approximation in which the third equation required for evaluating the coefficients d_i arises by assigning arbitrary values to the nose radius of curvature a given by equation (14). Dumitrescu derived second approximations to the Froude number for values of a in the range $a = 0{\cdot}5b$–$0{\cdot}9b$, demonstrating incidentally that there is an infinite number of approximate solutions which will satisfy the constant-pressure condition to this level of approximation. The appropriate nose curvature and hence the Froude number could have been selected by comparison with the shape which he observed experimentally but Dumitrescu adopted an alternative approach which has great appeal because it takes into account the influence of the gross features of the flow some way downstream on the flow near the stagnation point. His method thus provides a very good description of the overall geometry of the slug. It assumes that the nose shape is spherical with radius of curvature equal to that at the stagnation point and then selects the nose curvature which allows this shape to merge with the shape some way downstream where the liquid falls freely under gravity. A one-dimensional model of the flow in that region shows that the asymptotic form is given by the expression

$$\frac{x}{b} = \frac{Fr^2}{2[1-(r/b)^2]^2}. \tag{24}$$

The coefficients d_i have been recalculated† for this problem and found to be

$$d_1 = 0{\cdot}801090, \quad d_2 = 0{\cdot}138498, \quad d_3 = 0{\cdot}060412, \tag{25}$$

which give

$$U/(gb)^{\frac{1}{2}}_{\Omega=0} = 0{\cdot}496, \quad (a/b)_{\Omega=0} = 0{\cdot}75, \tag{26}$$

in agreement with Dumitrescu's solution.

The continuous lines in figures 2, 3 and 4 are produced by substituting the values given in (25) into (14) and (19) and the agreement with experiment is seen to be excellent. We note that by neglecting the dependence of the coefficients on Rossby number this approximate solution is strictly speaking a three-term second approximation only when $\Omega = 0$. One feature it has is that, in accordance with the assumption that $Ro > 0{\cdot}52$, the solution must be truncated at $\Omega^2 b/g = 34$

† Dumitrescu's coefficients are defined slightly differently and in fact the values finally obtained are not quoted in his paper. From the present solution, Dumitrescu's coefficients for $a/b = 0{\cdot}75$ are inferred to be $k_1 = -0{\cdot}10370$, $k_2 = -0{\cdot}00979$ and $k_3 = -0{\cdot}00295$.

when this value of Ross by number is achieved. This feature may possibly be brought about by ignoring the dependence of d_i on Rossby number.

4. Concluding remarks

Agreement with experiment is a pleasing feature of both approximate theories considered here and this gives confidence in the ideas on which they are based. Dumitrescu's method provides a neat *ad hoc* model for the solution to the free-boundary problem for the non-rotating situation. It successfully describes the flow near the stagnation point while incorporating the essential features of the flow downstream (see, for example, the comparison between the experimental and theoretical shape in Dumitrescu's figure 9). With the shape at $\Omega = 0$ determined in this way, the subsequent development of the shape with increasing Ω is then satisfactorily described through the quantity α_i appearing in (8). The major influence appears to come from the first term, which alone would provide a tolerable approximate description of the behaviour.

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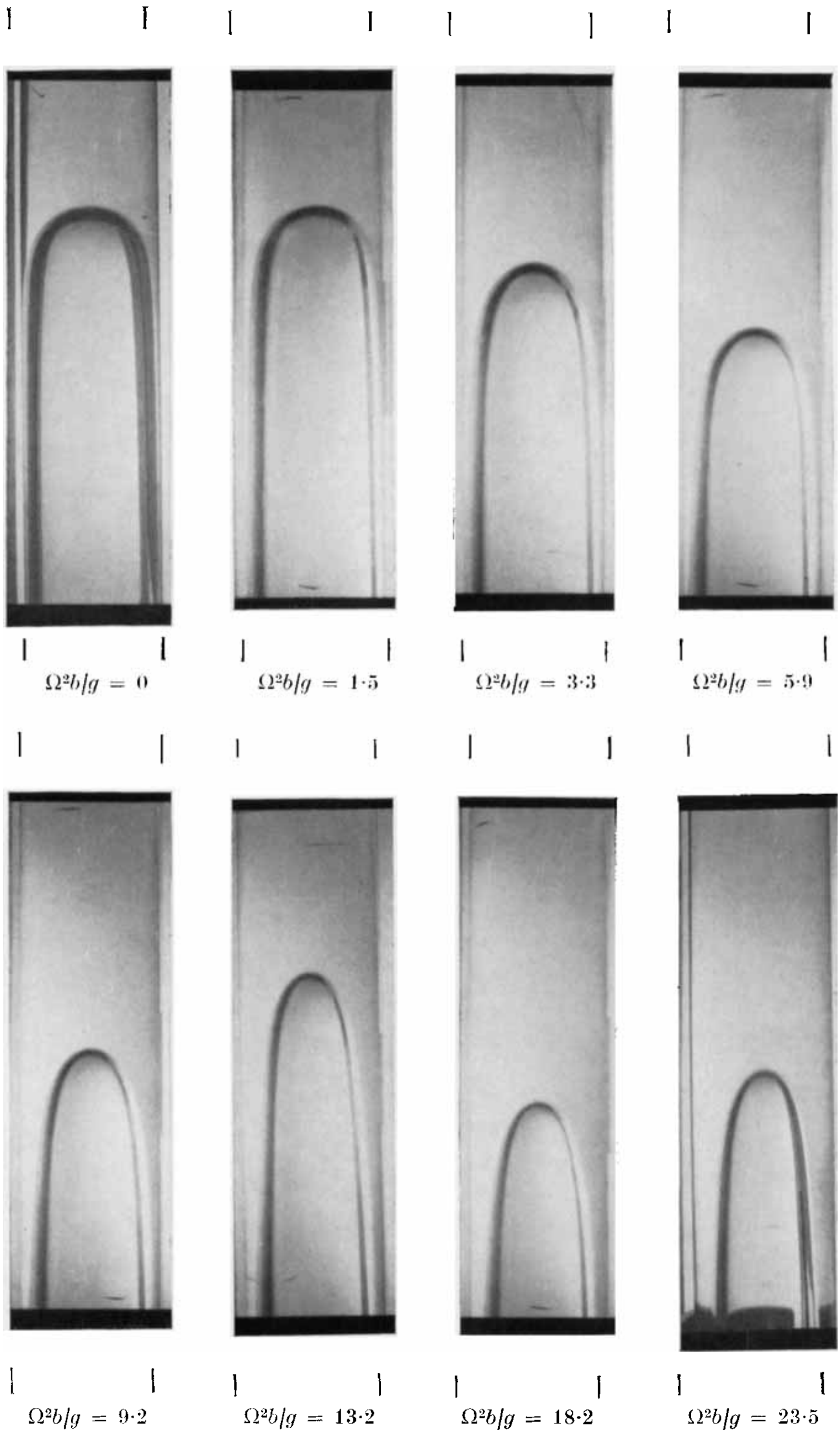

FIGURE 5. Bubble shapes at various values of $\Omega^2 b/g$. The lines above and below each photograph define the external dimensions of the tube.